JEE Main (Archive)	Daily Tutorial Sheet - 1

1.(D) $\int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $1+\frac{1}{x^4} = t^4 \Rightarrow \frac{-4}{x^5}dx = 4t^3dt \Rightarrow \frac{dx}{x^5} = -t^3dt$

Hence, the integral becomes $\int \frac{-t^3dt}{t^3} = -\int dt = -t + c = -\left(1+\frac{1}{x^4}\right)^{1/4} + c$

2.(B) $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx = \int e^{x+\frac{1}{x}}dx + \int x\cdot\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx = \int e^{x+\frac{1}{x}}dx + xe^{x+\frac{1}{x}} - \int\frac{d}{dx}(x)e^{x+\frac{1}{x}}dx$

$= \int e^{x+\frac{1}{x}}dx + xe^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}}dx \quad = \int e^{x+\frac{1}{x}}dx + xe^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}}dx = xe^{x+\frac{1}{x}} + c$

3.(C) Given, $\int f(x)dx = \psi(x)$

Let $I = \int x^5 f(x^3)dx$

Put $x^3 = t \quad \Rightarrow \quad x^2dx = \frac{dt}{3}$ .....(i)

$\therefore \quad I = \frac{1}{3}\int t.f(t)dt = \frac{1}{3}\left[t.\int f(t)dt - \int\left\{\frac{d}{dt}(t)\int f(t)dt\right\}dt\right]$ [integration by parts]

$= \frac{1}{3}\left[t\psi(t) - \int\psi(t)dt\right] = \frac{1}{3}\left[x^3\psi(x^3) - 3\int x^2\psi(x^3)dx\right] + c$ [from equation (i)]

$= \frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)dx + c$

4.(C) Let $I = \int \frac{\sin^2 x\cos^2 x}{(\sin^2 x+\cos^2 x)^2(\sin^3 x+\cos^3 x)^2}dx = \int\frac{\sin^2 x\cos^2 x\,dx}{(\sin^3 x+\cos^3 x)^2} = \int\frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)^2}dx$

Put $1+\tan^3 x = t$ so that $3\tan^2 x\sec^2 x\,dx = dt$

$\therefore \; I = \frac{1}{3}\int\frac{dt}{t^2} = -\frac{1}{3}\cdot\frac{1}{t} + C = -\frac{1}{3}\cdot\frac{1}{(1+\tan^3 x)} + C$

5.(A) Given, $f\left(\frac{x-4}{x+2}\right) = 2x+1$

Put $\frac{x-4}{x+2} = t \Rightarrow x-4 = 2t+xt \Rightarrow x-xt = 2t+4 \Rightarrow x(1-t) = 2(t+2) \Rightarrow x = \frac{2(t+2)}{1-t}$

$\therefore$ (i) becomes $f(t) = 2\left(\frac{2(t+2)}{1-t}\right)+1 = \frac{-4(t+2)}{t-1}+1$ or $f(x) = \frac{-4(x+2)}{x-1}+1$

on integrating (ii), we get $\int f(x) = -4\int\frac{(x+2)}{x-1}dx + \int 1\,dx$

$= -4\int\frac{x+3-1}{x-1}dx + x = -4\int 1\,dx - 12\int\frac{1}{x-1}dx + x = -3x + 12\int\frac{1}{1-x}dx = -3x + 12\log_e|1-x| + C$

6.(A) Let $I = \int \frac{2x+5}{\sqrt{7-6x-x^2}} dx \quad \Rightarrow \quad I = \int \frac{2x+6-1}{\sqrt{7-6x-x^2}} dx$

$\Rightarrow \quad I = \int \frac{2x+6}{\sqrt{7-6x-x^2}} dx - \int \frac{1}{\sqrt{(4)^2-(x+3)^2}} dx$

$\Rightarrow \quad I = \int \frac{-dt}{\sqrt{t}} - \int \frac{1}{\sqrt{4^2-(x+3)^2}} dx \qquad [\because t = 7-6x-x^2 \Rightarrow dt = -(2x+6)dx]$

$\Rightarrow \quad I = -2(7-6x-x^2)^{1/2} - \sin^{-1}\left(\frac{x+3}{4}\right) + C \qquad \Rightarrow \quad I = -2\sqrt{7-6x-x^2} - \text{sinh}-1\left(\frac{x+3}{4}\right) + C$

$\therefore \quad A = -2, \; B = -1$

7.(D) Let $I = \int \frac{\tan x}{1+\tan x+\tan^2 x} dx = \int \frac{1+\tan x - 1}{1+\tan x+\tan^2 x} dx = \int \frac{1+\tan x+\tan^2 x - \sec^2 x}{1+\tan x+\tan^2 x} dx$

$= \int \left(1 - \frac{\sec^2 x}{1+\tan x+\tan^2 x}\right) dx = x - \int \frac{\sec^2 x}{1+\tan x+\tan^2 x} dx$

$= x - \int \frac{1}{1+t+t^2} dt$ (Putting $\tan x = t \Rightarrow \sec^2 x dx = dt$)

$= x - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = x - \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{t+\frac{1}{2}}{\sqrt{3}/2}\right) + C = x - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x + 1}{\sqrt{3}}\right) + C \qquad \therefore \; K = 2, A = 3$

8.(A) Let $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx \qquad \ldots\text{(i)}$

Changing x to $-\pi/2 + \pi/2 - x = -x$, we have $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx \qquad \ldots\text{(ii)}$

Adding (i) and (ii), we get

$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x \left\{\frac{1}{1+2^x} + \frac{1}{1+2^{-x}}\right\} dx = \int_{-\pi/2}^{\pi/2} \sin^2 x = 2\int_0^{\pi/2} \sin^2 x \, dx$ (as the integral function is even)

$\therefore \quad I = \frac{\pi}{4}$

9.(A) Let $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \left(1 + \log\left(\frac{2+\sin x}{2-\sin x}\right)\right) dx = \int_{-\pi/2}^{\pi/2} \sin^4 x \, dx + \int_{-\pi/2}^{\pi/2} \sin^4 x \log\left(\frac{2\sin x}{2-\sin x}\right) dx = I_1 + I_2$

Now, $I_1 = \int_{-\pi/2}^{\pi/2} \sin^4 x \, dx = 2\int_0^{\pi/2} \sin^4 x \, dx \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(x) = f(-x)\right]$

$$I_2 = \int_{-\pi/2}^{\pi/2} \sin^4 x \log\left(\frac{2+\sin x}{2-\sin x}\right) dx = 0 \qquad \left[\because \int_{-a}^{a} f(x)dx = 0 \text{ if } f(-x) = -f(x)\right]$$

$$\therefore\ I = 2\int_0^{\pi/2} \sin^4 x = 2\int_0^{\pi/2} (\sin^2 x)^2\, dx = 2\int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{2}\int_0^{\pi/2} [1+\cos^2 2x - 2\cos 2x]\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right] dx = \frac{1}{2}\int_0^{\pi/2} \left(\frac{3}{2}+\frac{\cos 4x}{2} - 2\cos 2x\right) dx$$

$$= \frac{1}{2}\left[\frac{3}{2}|x|_0^{\pi/2} + \frac{1}{2}\left|\frac{\sin 4x}{4}\right|_0^{\pi/2} - 2\left|\frac{\sin 2x}{2}\right|_0^{\pi/2}\right] = \frac{1}{2}\left[\left(\frac{3}{2}\times\frac{\pi}{2}\right) + \frac{1}{8}(\sin 2\pi - \sin 0) - \sin\pi + \sin 0\right] = \frac{3\pi}{8}$$

10.(D) Let $I = \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx \quad \Rightarrow I = \int_{\pi/4}^{3\pi/4} \left(\frac{x}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}\right) dx$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{x(1-\sin x)}{\cos^2 x} dx \quad \Rightarrow I = \int_{\pi/4}^{3\pi/4} x\cdot\sec^2 x\, dx - \int_{\pi/4}^{3\pi/4} x\cdot\frac{\sin x}{\cos^2 x} dx$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} x\cdot\sec^2 x\, dx - \int_{\pi/4}^{3\pi/4} x(\sec x\tan x) dx$$

$$\Rightarrow I = |x\tan x|_{\pi/4}^{3\pi/4} - \log(\sec x)|_{\pi/4}^{3\pi/4} - \left|\frac{x}{\cos x}\right|_{\pi/4}^{3\pi/4} - \int_{\pi/4}^{3\pi/4} \frac{1}{\cos x} dx$$

$$\Rightarrow I = |x\tan x|_{\pi/4}^{3\pi/4} - \left|\frac{x}{\cos x}\right|_{\pi/4}^{3\pi/4} - \log|\sec x + \tan x|_{\pi/4}^{3\pi/4}$$

$$\Rightarrow I = -\frac{3\pi}{4} - \frac{\pi}{4} + \frac{3\pi}{4}(\sqrt{2}) + \frac{\pi}{4}\sqrt{2} - \log(\sqrt{2}+1) + \log(\sqrt{2}+1) \Rightarrow I = -\pi + \sqrt{2}\pi \Rightarrow I = \pi(\sqrt{2}-1)$$

11.(AC) Let $x^2 - 1 = y$

Then, the given integral becomes

$$\int \frac{dy}{2}\sqrt{\frac{2\sin y - \sin 2y}{2\sin y + \sin^2 y}} = \frac{1}{2}\int dy\ \tan\frac{y}{2} = \frac{-\frac{1}{2}\log\left(\cos\frac{y}{2}\right)}{\frac{1}{2}} + c = \log\left|\sec\left[\frac{(x^2-1)}{2}\right]\right| + c$$

12.(C) If $f(x) = \int \frac{5x^8+7x^6}{\left(x^2+1+2x^7\right)^2} dx = \int \frac{\frac{5}{x^6}+\frac{7}{x^8}}{\left(2+\frac{1}{x^7}+\frac{1}{x^5}\right)^2} dx$

Put $\quad 2+\frac{1}{x^7}+\frac{1}{x^5} = t$

$$\left(\frac{-7}{x^8}-\frac{5}{x^6}\right)dx = dt \ ; \quad -\int\frac{dt}{t^2}=\frac{1}{t}; \quad f(x)=\frac{1}{2+\frac{1}{x^7}+\frac{1}{x^5}}+C \ ; \ f(0)=0=C \Rightarrow f(1)=\frac{1}{4}$$

13.(C) $I = \int \frac{\left(\sin^n\theta-\sin\theta\right)^{\frac{1}{n}}\cos\theta\, d\theta}{(\sin\theta)^{n+1}}$

$$I = \int\left(1-\frac{1}{(\sin\theta)^{n-1}}\right)^{\frac{1}{n}}\frac{\cos\theta}{(\sin\theta)^n}\,d\theta$$

Let $1-\frac{1}{(\sin\theta)^{n-1}} = x$

$$\frac{\cos\theta d\theta}{(\sin\theta)^n}=\frac{dx}{n-1} \qquad \therefore\ I = \int x^{\frac{1}{n}}\,dx \times \frac{1}{n-1}$$

$$\frac{x^{\frac{1}{n}+1}}{\left(\frac{1}{n}+1\right)(n-1)}+C = \frac{n\,x^{(n+1)/n}}{n^2-1}+C = \frac{n}{n^2-1}\left\{1-\left(\frac{1}{\sin\theta}\right)^{n-1}\right\}^{\frac{n+1}{n}} + C$$

14.(D) Let $-4x^3=t$

$$-12x^2dx = dt \qquad \therefore\ \int x^5e^{-4x^3}dx = \frac{1}{48}\int te^t = \frac{1}{48}e^t(t-1) = \frac{1}{48}e^{-4x^3}(-4x^3-1)$$